A BRIEF INTRODUCTION TO RIEMANN SURFACES

MATTHEW HUYNH

Abstract. This paper aims to give a friendly introduction Riemann surfaces. We begin with some standard facts from complex analysis, and then give the definition and examples of Riemann surfaces. Finally we introduce the notion of the Riemann surface of a holomorphic function, which is the historical motivation for studying them. We will follow Donaldson's book *Riemann Surfaces* [1].

Contents

1. A complex analysis refresher

Throughout, Ω will denote a connected open subset of $\mathbb{C}$ commonly called a domain.

Definition 1.1 (Holomorphic function)**.** Let $f : \Omega \to \mathbb{C}$ be a function. We call f holomorphic at a point $z_0 \in \Omega$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

converges in $\mathbb{C}$. If f is holomorphic at z_0 for all $z_0 \in \Omega$, then we call f holomorphic on Ω and write $f \in \mathcal{H}(\Omega)$.

If we forget that $\mathbb{C}$ is a field, then we can identify it with $\mathbb{R}^2$ and we get a notion of differentiability of functions between open sets in linear spaces: if $f : \Omega \to \mathbb{C}$ is a continuous function, we call it differentiable at $z_0 \in \Omega$ if there exists a real-linear map $T : \mathbb{C} \to \mathbb{C}$ satisfying

$$\frac{|f(z_0 + h) - f(z_0) - T(h)|}{|h|} \to 0, \qquad \text{as } |h| \to 0$$

A holomorphic function is differentiable, but the criterion in 1.1 is stricter than the one we just gave. It says that the map T, called the differential, is multiplication by a complex number w. If we use the coordinates (x, y) where $z = x + iy$ for $\mathbb{C}$,

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then with these coordinates, the differential at z_0 for a holomorphic function takes the form

$$T_{z_0} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $r = |w|$ and $\theta = \arg(w)$. The picture geometrically is that the differential of a holomorphic function is given by rotation composed with dilation. An example of a differentiable map that is not holomorphic is the map $z \mapsto \overline{z}$. Some examples of holomorphic functions include polynomials, the exponential, and the trig functions.

You may ask yourself: given a holomorphic function f on Ω, can I extend it to be holomorphic on a bigger set? Such a question is made precise using the notion of analytic continuation. (Such a process might be called "holomorphic continuation", however the term "analytic" was used first historically which is not a problem, since a function is holomorphic if and only if it is analytic [2]).

Definition 1.2 (Analytic continuation)**.** Let f be a holomorphic function defined on a neighborhood U_0 of $z_0 \in \mathbb{C}$. Let γ be a path starting at z_0. An analytic continuation of f along γ is a family of holomorphic functions f_t for $t \in [0,1]$ where f_t is defined on a neighborhood U_t of $\gamma(t)$ such that

- $f_0 = f$ on some neighborhood of z_0;
- for each $s \in [0,1]$ there is a $\delta_s > 0$ such that if $|t - s| < \delta_s$, then the functions f_t and f_s are equal on their common domain $U_s \cap U_t$.

As an example, we will consider the complex logarithm. If $z = re^{i\theta}$, then a sensible definition of the logarithm is $\log(z) = \log(r) + i\theta$ where the logarithm on the right-hand side is the log of a real variable. This definition satisfies $e^{\log(z)} = \log(e^z) = z$ however, z also equals $re^{i(\theta+2\pi k)}$ for all $k \in \mathbb{Z}$, so this function is not well-defined on all of $\mathbb{C}$. If we make what is called a "branch cut" along the negative real axis, then on $\mathbb{C} \setminus \mathbb{R}^{\leq 0}$ we can define the complex logarithm as above stipulating that θ takes values in $(-\pi, \pi)$ to get an actual function. We did not have to choose the negative real axis and the interval (π, π); we could have chosen to cut out any ray starting at the origin and an appropriate interval $(a, a + 2\pi)$.

Here's where analytic continuation comes in. Let f be the complex logarithm with branch cut $\mathbb{R}^{\leq 0}$ and $\theta \in (-\pi, \pi)$ defined on $B_1(0)$. Let $\gamma(t) = e^{2\pi i t}$ which is a loop around the origin. Then we can analytically continue f along γ: f_t will be the "logarithm" defined on $B_1(\gamma(t))$, but with branch cut $\mathbb{R}^{\leq 0} \cdot e^{2\pi i t}$. We can picture this as a ball moving along the unit circle, and the branch cut moving along with the center of the ball at the same speed. Doing this forces us to end up with $f_1(z) = \log(r) + i\theta$ with θ taking values in $(\pi, 3\pi)$.

If the fact that the logarithm is not globally defined on $\mathbb{C}$ bothers you, do not worry. We will soon introduce the necessary tools to produce a Riemann surface on which the logarithm is a globally defined function to $\mathbb{C}$.

2. Definition and examples of Riemann surfaces

Definition 2.1 (Riemann surface)**.** A Riemann surface X is the following data: a Hausdorff topological space X and an equivalence class of an atlas $\{(U_i, \phi_i)\}_{i \in I}$. Here each U_i is an open set in X and ϕ_i is a homeomorphism between U_i and an open subset U_i' of $\mathbb{C}$. We require that the collection of U_i cover X, and that for each $i, j \in I$, the transition function $\phi_j \circ \phi_i^{-1}$ from $\phi_i(U_i \cap U_j)$ to $\phi_j(U_i \cap U_j)$ is a holomorphic function.

Two atlases $\{(U_i, \phi_i)\}_{i\in I}$ and $\{(V_j, \psi_j)\}_{j\in J}$ for X are equivalent if their disjoint union is again an atlas for X. The functions ϕ_i are called coordinate charts, or just charts for short.

Example 2.2. An open set Ω in $\mathbb{C}$ is a Riemann surface with the atlas $\{\Omega, \mathrm{id}_\Omega\}$. Some important instances of these are the unit disk $D = \{|z| < 1\}$ and the upper-half plane $H = \{\mathrm{Im}(z) > 0\}$. These are equivalent Riemann surfaces under the Cayley map $T : H \to D$ given by

$$z \mapsto \frac{z-i}{z+i}$$

Example 2.3. Another example is the Riemann sphere S^2. As a set, this is just $\mathbb{C}$ with the extra point $\{\infty\}$. The topology is generated by open sets in $\mathbb{C}$ together with $\{\infty\} \cup (\mathbb{C} \setminus K)$ where K is a non-empty compact subset of $\mathbb{C}$. We can make S^2 into a Riemann surface with the following atlas:

$$U_0 = \{|z| < 2\}, \quad U_1 = \{|z| > 1/2\}$$
$$\phi_0 = \mathrm{id}_{U_0}, \quad \phi_1(z) = 1/z$$
$$U_0' = U_1' = U_0$$

The transition functions $\phi_0 \circ \phi_1^{-1}$ and $\phi_1 \circ \phi_0^{-1}$ are both the function $z \mapsto 1/z$ from the annulus $\{1/2 < |z| < 2\}$ to itself. Hence S^2 is a Riemann surface.

Here is an example for the algebraic folks if you know some differential topology.

Example 2.4. Let $P(z, w)$ be a complex polynomial in two variables and let $X \subset \mathbb{C}^2$ be the set of points (z_0, w_0) for which $P(z_0, w_0) = 0$. If for all $(z_0, w_0) \in X$ one of

$$\frac{\partial P}{\partial z}(z_0, w_0) \quad \text{or} \quad \frac{\partial P}{\partial w}(z_0, w_0)$$

is non-zero, then X is a Riemann surface. This follows from the implicit function theorem in the holomorphic case and our assumption that 0 is a regular value.

Next up on the docket are maps between Riemann surfaces. Of course, we could talk about any old maps, but here is the notion of a holomorphic map between Riemann surfaces.

Definition 2.5. Let X and Y be Riemann surfaces with atlases $\{(U_i, \phi_i)\}_{i\in I}$ and $\{(V_j, \psi_j)\}_{j\in J}$ respectively. Then a map $F : X \to Y$ is called holomorphic if for each i and j, the composition

$$\psi_j \circ F \circ \phi_i^{-1} : \phi_i(U_i \cap F^{-1}(V_j)) \to V_j'$$

is holomorphic in the sense of 1.1.

Two Riemann surfaces X and Y are isomorphic if there exists a holomorphic bijection $F : X \to Y$ whose inverse is also holomorphic.

3. The Riemann surface of a holomorphic function

Finally, here are the promised goods. The analytic continuation of a function defined on a Riemann surface X along a path is defined analogously as in 1.2, just replacing $\mathbb{C}$ with X.

Theorem 3.1. *Let Y be a connected Riemann surface, y_0 a point in Y, and ψ_0 a holomorphic function defined on a neighborhood of y_0. Then there exists a Riemann surface X, a locally homeomorphic holomorphic function $F : X \to Y$, a point $x_0 \in F^{-1}(y_0)$, and a holomorphic function Ψ on X such that*

- *ψ_0 can be analytically continued along a path γ in Y if and only if γ has a lift $\tilde{\gamma}$ in X starting at x_0*
- *The analytic continuation of ψ_0 along γ has ψ_1 equal to $\Psi \circ F_{\tilde{\gamma}}^{-1}$ in a neighborhood of $\gamma(1)$.*

Proof. Define X' to be the set of pairs (γ, ψ_t) where γ is a path in Y based at y_0, and ψ_t is an analytic continuation of ψ_0 along γ (recall from 1.2 that an analytic continuation is given by a one-parameter family of holomorphic functions). Define X to be X' modulo the following equivalence relation: $(\gamma, \psi_t) \sim (\tilde{\gamma}, \tilde{\psi}_t)$ if and only if $\gamma(1) = \tilde{\gamma}(1)$ and ψ_1 and $\tilde{\psi}_1$ agree on a neighborhood of $\gamma(1)$ in Y. Let $F : X \to Y$ be given by $(\gamma, \psi_t) \mapsto \gamma(1)$, and let $\Psi : X \to \mathbb{C}$ be given by $(\gamma, \psi_t) \mapsto \psi_1(\gamma(1))$.

Now we check that X, F, and Ψ do what we want them to do. Our first task is to topologize X. Let $(\gamma, \psi_t) \in X$. Then $\gamma(1)$ lies in the domain of a chart U, and without loss of generality we can take U to be homeomorphic to a disk in $\mathbb{C}$. Without loss of generality, the domain V of ψ_1 can be taken to be homeomorphic to a disk as well. Then define $U' \subset X$ to be the set of (γ', ψ'_t) for which $\gamma'(1) \in U \cap V$ and ψ'_1 agrees with ψ_1 inside $U \cap V$. We declare that U' is homeomorphic to $U \cap V$, and we let these U' generate the topology on X.

X is Hausdorff essentially because Y is, and this topology makes F a local homeomorphism. As for charts, if (γ, ψ_t) in X is contained in U' as above, use the chart ϕ_i for U pulled back along F. The condition that the transition functions are holomorphic follow from the fact that F is a local homeomorphism and the fact that this condition holds for charts for Y. So X is a Riemann surface. By the same reasoning, F is holomorphic, because it is just the identity map in charts. Ψ is holomorphic because in charts, it is given by a holomorphic function ψ_1.

Finally we check the two bullet points above. Let $x_0 = (y_0, \psi_0)$ where y_0 denotes the constant path $[0, 1] \to y_0$. If γ has a lift to X starting at x_0, then ψ_0 can be analytically continued along γ because of how we have defined X. On the other hand, if ψ_0 can be analytically continued along a path γ, then its lift $\tilde{\gamma}$ in X can be constructed as follows: at time $s \in [0, 1]$, $\tilde{\gamma}(s) = (\gamma_s, \psi_{st})$ where γ_s is γ pull backed by $x \mapsto sx$ for $x \in [0, 1]$. As for the second item, let ψ be the analytic continuation of ψ_0 along γ. If we consider the lift $\tilde{\gamma}$ and the point (γ_1, ψ_t) lying above $\gamma(1)$, then ψ_1 equals ψ on a neighborhood of $\gamma(1)$, and so ψ equals $\Psi \circ F_{\tilde{\gamma}}^{-1}$ on that neighborhood. □

Example 3.2. Recall our discussion of the logarithm earlier. We could have continued f defined above around a circle that wound around the origin n times for $n \in \mathbb{Z}$, with $-n$ denoting going around the origin clockwise. We also could have started at any point in $\mathbb{C}^*$ and resized the balls accordingly. Thus it does not take much to convince yourself that the Riemann surface associated to the logarithm can be visualized as a surface spiraling down the z-axis in $\mathbb{R}^3$. You can think of going up or down a level as adding or subtracting $2\pi i$ from the value of the logarithm.

Example 3.3. Another illustrative example is the complex square root function. In $\mathbb{C} \setminus \{0\}$, each number has two square roots, so the square root function is not globally defined. However, we can make use of branch cuts to define the square root on say $\mathbb{C} \setminus \mathbb{R}^{\leq 0}$. It is given by $\sqrt{z} = |z|^{1/2} e^{i\theta/2}$ for $\theta \in (-\pi, \pi)$. We can analytically continue this function defined on $B_1(1)$ around the origin in a circle again, although this time, going around the origin once gives us a minus sign since θ now takes values in $(\pi, 3\pi)$, and going around twice gives us back our original function since θ now takes values in $(3\pi, 5\pi)$. The Riemann surface associated to the square root (say defined on a neighborhood of the unit disc) can be visualized as follows:

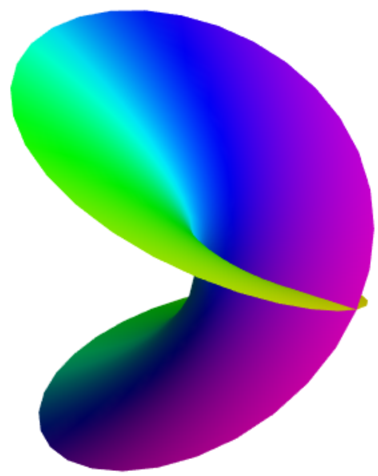

FIGURE 1. The Riemann surface associated to the square root function

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References

[1] Simon Donaldson. Riemann Surfaces. Oxford University Press. 2011.
[2] Wilhelm Schlag. A Course in Complex Analysis and Riemann Surfaces. American Mathematical Society. 2014.